Solving conic section problems using Celestial Mechanics

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Abstract

In mathematics, a conic section is a curve obtained when a plane intersects a double cone. Depending on the angle of intersection, three types of conic sections are possible: namely hyperbola, parabola and ellipse (circle is a special case of an ellipse). In real life, conic sections arise most commonly in celestial orbits - e.g., Earth rotates around the Sun in an elliptical orbit. In fact, celestial objects like planets, comets, asteroids all follow the conic loci under the influence of the laws of gravitation. Most problems in conic sections are solved using coordinate geometry, a popular mathematical technique. In this article, we show an alternate method to solve a complex conic section problem related to finding the angle of intersection between a circle and a parabola, using celestial mechanics.

1 Finding angle of intersection between a circle and a parabola with common focus

Consider a circle with equation

$$(x-a)^2 + (y-b)^2 = d^2 \qquad \text{(center is } (a,b) \text{ and radius is } d\text{)}$$

and a parabola with equation

$$(x-a)^2 + b^2 - c^2 = 2(b-c)y \qquad \text{(focus is } (a,b) \text{ and directrix is } y=c\text{)}$$

satisfying the additional condition $|b-c| < 2d$ and $c < b$ (this ensures the conics intersect).

Find the angle between the tangents drawn to the circle and parabola at any of the intersection points.

How do we usually do coordinate geometry problems?

1. Graph the circle and parabola to get an idea of how the curves look like.
2. Find the coordinates of the intersection points, using algebra.
3. Next, we find the derivative of both the conics at the intersection points. This is required to find the slope of the tangents.
4. Lastly, we find the angle between these tangent lines using trigonometry.

In summary, this problem can be solved using a combination of algebra, trigonometry and a bit of calculus. Is there any alternative way of solving this using celestial mechanics? The answer is Yes. Let's see how.

First, notice that the focus of both the conics are (a, b).

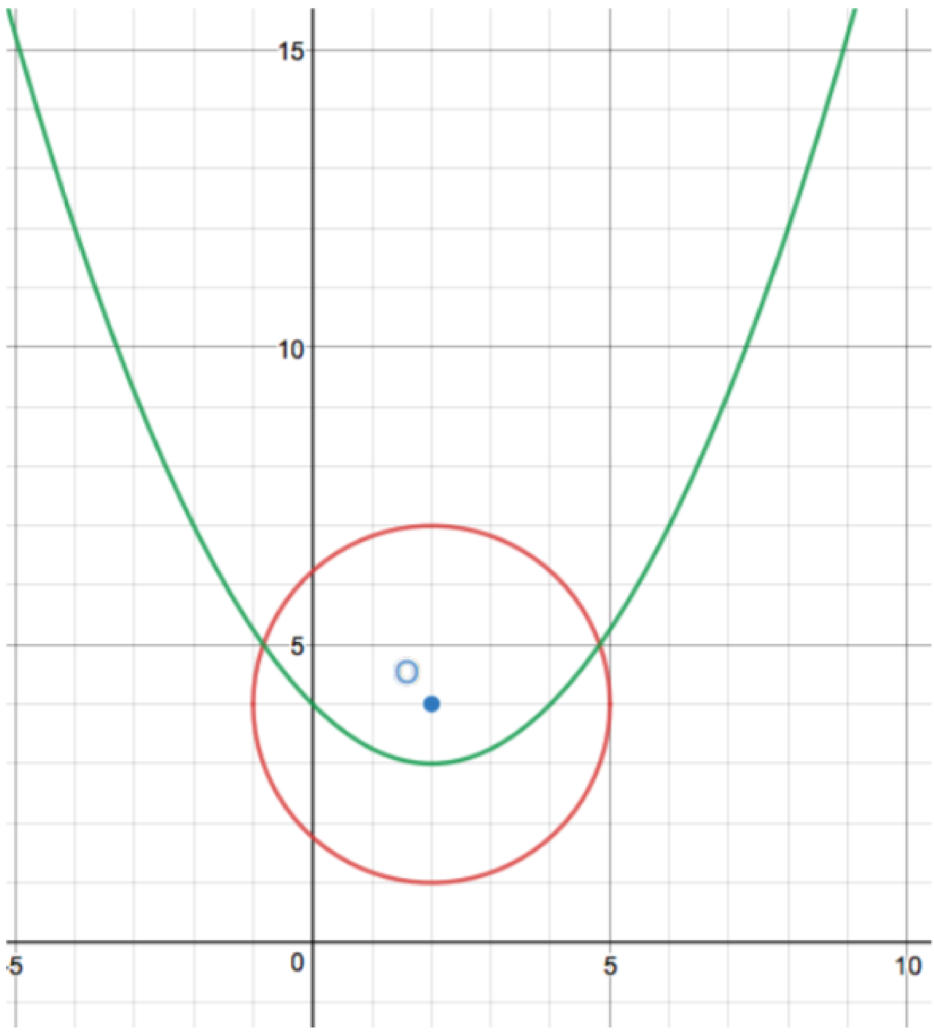

Figure 1: Circle and parabola with same center/focus

This motivates us to consider these conics as orbits of celestial objects moving under the influence of Gravitational force that Sun exerts on it.

Let us summarise what this means for a circle and parabola having same center/focus, in terms of celestial mechanics.

- Conics $\equiv$ orbits of say planet and asteroid.
- Common focus $\equiv$ Sun (or any object with mass $\gg$ mass of planet and asteroid) around which the planet and asteroid revolve.

We are getting into astronomy. To be precise, celestial mechanics. Let's revise what we know from basic gravitation.

2 Gravitation Laws

We state the laws which will be relevant in solving the original problem without proof assuming they are well-known. We assume the mass of central object as M and mass of orbiting object as m where $M \gg m$.

- Gravitational force exerted at a distance r is
$$\frac{GMm}{r^2}.$$
- Gravitational Potential Energy possessed by orbiting body at a distance r is
$$P.E. = -\frac{GMm}{r}$$
and Kinetic Energy possessed by orbiting body is
$$K.E. = \frac{1}{2}mv^2.$$
- Conservation of total mechanical energy means $K.E. + P.E.$ is constant.
- Conservation of angular momentum states that $m \cdot (\vec{r} \times \vec{v})$ is constant for the orbiting body.
- Nature of orbit: If total mechanical energy < 0 then the body follows an elliptical orbit. If total mechanical energy is 0 then parabola and if total mechanical energy > 0 then hyperbolic orbit.

3 Solving the problem using Celestial Mechanics

We need to rephrase the coordinate problem into a physics/astronomy problem.

Problem 1: Consider a planet "Earth-A" with mass m_1 orbiting a star "Sun-A" with mass M in a perfectly circular orbit of radius R. A comet with mass m_2 from far away approaches the Sun-A and due to the gravitational force follows a parabolic path. The comet's approach to Sun-A is closest at a distance r Assume that there are no other external forces on this system. Assume that the orbit of the comet cuts Earth-A's orbit at two points. Also assume that $M \gg m_1, m_2$.

Find the velocity of the comet when it crosses Earth-A's orbit in the direction tangent to comet's orbit and in the direction tangent to Earth-A's orbit and angle between these vectors.

How do we start this problem? We'll first notice what all can be derived quickly.

We need to find the velocity of comet in direction tangent to comet's orbit. This is essentially the speed of the comet. Thus we need to find the K.E of the comet when it crosses Earth-A's orbit. Finding K.E. isn't the best thing when we are given the distance. But we can find P.E quickly if we know the distance to the comet from Sun-A. If we can get total mechanical energy, we can then find K.E also! Well we have got a logic to find the velocity of comet in the direction tangent to it's orbit when it crosses Earth-A's orbit.

Let's find the potential energy. When the comet crosses Earth-A's orbit, it is at a distance R from Sun-A. Thus the potential energy is

$$P.E. = -\frac{GMm_2}{R}.$$

We know that the total mechanical energy in a parabolic orbit is 0. Thus we have $K.E. + P.E. = 0$. Let v_1 be the velocity of comet in the direction tangent to it's orbit when it crosses Earth-A's orbit. Thus we have

$$K.E. + P.E. = 0 \implies \frac{1}{2}m_1 v_1^2 - \frac{GMm_1}{R} = 0 \implies v_1 = \sqrt{\frac{2GM}{R}}.$$

v_1 is actually the *escape velocity* (as in a parabolic path, object reaches infinity with $K.E. = 0$). We are done with first part of the problem.

Coming to the first part, we need to find the velocity of the comet when it crosses Earth-A's orbit in the direction tangent to Earth-A's orbit. What's so special about Earth-A's orbit? We don't wanna use part 1 to find part 2 (by finding angles, etc which again involves mathematics and not physics). We know that Earth-A's orbit is circular! So we have tangent to circle. Okay, wait. Tangent to circle is perpendicular to radial vector at that point. Where do we relate velocity vector and radial vector? Angular momentum!

We need to find the angular momentum of the comet. We know the closest approach of comet is at a distance r from Sun-A. At this instance the velocity vector and radial vector are perpendicular. Thus we have angular momentum L as

$$L = m(\vec{r} \times \vec{v}) = mvr.$$

We need to find out v. As the comet follows a parabolic path, the total mechanical energy of the comet is 0.

$$v = \sqrt{\frac{2GM}{r}} \implies L = mr \cdot \sqrt{\frac{2GM}{r}} = m\sqrt{2GMr}.$$

Now at the intersection of comet's and planet's orbit, the velocity vector in direction tangent to circle will be perpendicular to radial vector. Let v_2 be the velocity of comet in direction tangent to Earth-A's orbit. Thus we have

$$L = m(\vec{r} \times \vec{v}) = mv_2 R \implies mv_2 R = m\sqrt{2GMr} \implies v_2 = \sqrt{\frac{2GMr}{R^2}}$$

and we are done with part 2 of the problem.

Now we are only left to find the angle between $\vec{v_1}$ and $\vec{v_2}$. Notice that $\vec{v_2}$ was a component of $\vec{v_1}$. Suppose the angle between $\vec{v_1}$ and $\vec{v_2}$ is θ, we have

$$v_2 = v_1 \cos\theta \implies \cos\theta = \frac{v_2}{v_1} = \frac{\sqrt{\frac{2GMr}{R^2}}}{\sqrt{\frac{2GM}{R}}} = \sqrt{\frac{r}{R}}.$$

Thus

$$\theta = \cos^{-1}\sqrt{\frac{r}{R}}.$$

In the problem we were given that the comet's orbit cuts Earth-A's orbit at two points thus $R > r$ or $\sqrt{\frac{r}{R}} < 1$ thus $\cos^{-1}\sqrt{\frac{r}{R}}$ exists. Final answer:

$$\boxed{\theta = \cos^{-1}\sqrt{\frac{r}{R}}}.$$

4 Solving the coordinate problem using the tools we developed

How do we relate the above variables in the original problem? Note equation of circle is similar to orbit of Earth-A and second equation of parabola is similar to orbit of comet. We have similarity between

$$R \equiv d$$

Shortest distance of comet or $r \equiv$ distance between focus and vertex of the parabola.

Now we are ready to solve the problem. We state the problem we had to solve initially.

Problem 2: Consider a circle with equation

$$(x-a)^2 + (y-b)^2 = d^2$$

and a parabola with equation

$$(x-a)^2 + b^2 - c^2 = 2(b-c)y$$

satisfying the additional condition $|b-c| < 2d$ and $c < b$.

Find the angle between the tangents drawn to the circle and parabola at one of the intersection point.

We need to find the angle between the lines. Firstly we make it simpler by shifting (a, b) to $(0, 0)$. As we have a left shift by a and down shift by b, we have to plug x as $x + a$ and y as $y + b$ in the original equations.

$$x^2 + y^2 = d^2$$

and

$$x^2 + b^2 - c^2 = 2(b-c)(y+b) = 2(b-c)y + 2b(b-c).$$

We also have $|b-c| < 2d$ or $b-c < 2d$. Now minimum distance between the parabola and focus is the distance between focus and vertex of parabola. Let the vertex of parabola be $(0, y_1)$. Plugging this in the equation gives

$$y_1 = \frac{b^2 - c^2 - 2b^2 + 2bc}{2(b-c)} = \frac{-(b-c)^2}{2(b-c)} = -\frac{b-c}{2}.$$

Now from *Problem 1*, angle between the tangents iis

$$\theta = \cos^{-1}\sqrt{\frac{r}{R}} = \cos^{-1}\sqrt{\frac{\frac{b-c}{2}}{d}}.$$

As $b-c < 2d$, $\frac{b-c}{2d} < 1$, thus θ exists. Final answer is

$$\boxed{\cos^{-1}\sqrt{\frac{b-c}{2d}}}.$$

Problems for the reader:

- Can we extend this problem to hyperbola and circle?
- Can we extend the problem to any pair of conics (circle, ellipse, parabola, ellipse) having the same focus?